The Hotelling Model of Natural Resource Extraction: A Primer

A supplement to the textbook
Elements of Environmental Management
by Werner Antweiler

The workhorse of resource economics is the **Hotelling model**. It makes use of optimal control theory, a mathematical tool for dealing with optimization problems that involve a long or infinite time horizon. Such models are usually written in continuous-time rather than discrete-time notation. Therefore, the mathematical expressions below are functions of time; e.g., $q(t)$ is the resource extracted from a mine at time t. Subscripts are used to denote derivates; e.g., c_s is the partial derivative of the cost function c with respect to the resource stock s. Dot notation is used to denote time derivatives. For example, $\dot{p}$ indicates price changes.

Consider a mining operation in a competitive market. In each time period t, the mining operator extracts a volume $q(t)$ from the mine and sells it at market price $p(t)$. The amount extracted in each period reduces the stock $s(t)$. In continuous-time notation, a dot on a variable denotes the derivative (change) with respect to time. In the simplest case, $\dot{s}(t) = -q(t)$; resources extracted from a known stock reduce that stock by that amount. Extracting the resource incurs a total cost of $c(t)q(t)$, where $c(t)$ is the marginal cost of resource extraction. In addition to the remaining stock $s(t)$, it is also useful to keep track of the discovered stock $f(t)$ ('finds'). In the simplest case of a fixed known stock, $f(t)$ is constant, but it may increase over time in the presence of resource exploration.

The presence of **resource exploration** is an essential feature of modern versions of the Hotelling model. Suppose that the mining company spends $x(t)$ on exploration, and that this expense generates $d(x, f)$ of new discoveries. Spending more will generate more discoveries ($d_x > 0$) but at a diminishing rate ($d_{xx} < 0$). The rate of discoveries also depends on the previously discovered stock f. The cumulative discoveries evolve with a motion equation $\dot{f}(t) = d(x(t), f(t))$. Any new discoveries $d(x, f)$ simply add to the inventory of what has been discovered so far. The change of the resource stock over time can then be described as $\dot{s}(t) = \dot{f}(t) - q(t)$. The known stock at a given point in time increases by new discoveries ($\dot{f}$) and diminishes by the amount extracted in each period (q). The difference $f(t) - s(t)$ is the total consumed (extracted) stock.

To make the model mathematically tractable, it is useful to introduce a particular functional form for the marginal cost:

$$c(t) = c_0 \left[\frac{f(t)}{s(t)}\right]^b \exp(-at) \tag{1}$$

Technological progress proceeds at a constant rate a and continues to lower the marginal extraction cost over time. The marginal extraction cost increases as the remaining stock diminishes. At the start when $s(0) = f(0) = s_0$, the marginal extraction cost is c_0 and then rises continuously and progressively as the remaining stock is depleted and the ratio of found to remaining stocks (f/s) increases. The constant $b > 1$ determines the progressivity of this cost increase and indicates the effect of **resource depletion**. Mining companies will first mine the cheapest resource before turning towards the more expensive resources. By totally differentiating equation (1) with respect to time, the rate of cost increases turns out to be

$$\begin{aligned} \frac{\dot{c}}{c} &= b\left[\frac{\dot{f}}{f} - \frac{\dot{s}}{s}\right] - a \\ &= b\left[\frac{q}{s} - \left(\frac{d}{f}\right)\left(\frac{f-s}{s}\right)\right] - a \end{aligned} \tag{2}$$

where time arguments have been dropped for notational convenience.

The right-hand side of this expression can be positive or negative. It can be negative when technological progress is rapid and a is large. The expression in square brackets tends to be positive, and is always positive in the absence of resource exploration (when $d = 0$).

The faster the mine is depleting its stock, the quicker it gets to the more expensive parts of the stock, and thus the faster its marginal cost rises. The presence of resource exploration dampens this cost increase. The two expressions in round parentheses are both positive: the ratio of new discoveries to total discoveries (d/f), and the ratio of used to unused stock ($(f-s)/s$). The first ratio will decrease over time, while the second ratio will increase over time. If resource exploration is sufficiently fast, the product of the two ratios can be larger than q/s, the extraction rate. In that case the right-hand side of (3) can be negative even when technological progress is slow.

The motion equation (3) provides important insights. The marginal cost of resource extraction does not necessarily have to increase continuously over time. Instead, it may follow a U shape. Especially at early stages of resource extraction, the beneficial effects of resource exploration and technological progress can lower the marginal cost. Only when **resource scarcity** becomes dominant at late stages in the mining operation will the marginal cost have to increase in pursuit of the more expensive parts of the remaining resource stock. This U-shape of the path of marginal costs over time is a very realistic feature of natural resource exploration. As we see below, this U-shape also emerges for the path of the resource price.

The cost function (1) has some further properties worth exploring. The dependence of the marginal cost function on the resource stock is defined by $c_s \equiv \partial c(s)/\partial s$, which in the case of (1) is given by $-(b/s)c$. Another way of putting this is that the elasticity of the marginal cost with respect to the resource stock is constant at $-b$. Depleting the resource stock by 1% increases the marginal extraction cost by b%.

The building blocks are now in place to look at the mining operation's profit maximization problem. Profits in each period t are revenue minus extraction and exploration costs:

$$\pi(t) = p(t)q(t) - c(t)q(t) - x(t) \tag{3}$$

Intertemporal profit maximization determines a *path* for the operational variables of the firm. Specifically, the mining operator's objective is to choose a path for extraction $q(t)$ and a path for exploration

$x(t)$ in order to maximize the time-discounted profit stream

$$\max_{q(t),x(t)} \int_{t=0}^{\infty} \pi(t) \exp(-rt) \mathrm{d}t \tag{4}$$

subject to the resource extraction constraint and a number of initial and terminal conditions. The optimal solution is obtained by maximizing the current-value Hamiltonian

$$H \equiv \pi(t) + \lambda(t)\dot{s}(t) + \mu(t)\dot{f}(t) \tag{5}$$

with respect to the two control variables (q and x) and the two state variables (s and f). The Hamiltonian introduces a shadow price on the resource constraint, $\lambda(t)$, and a shadow price on the exploration constraint, $\mu(t)$. The first shadow price captures the profitability of extracting an additional resource unit, and the second shadow price captures the profitability of discovering an additional resource unit. Maximizing (5) produces four first-order conditions (suppressing the time arguments for convenience of exposition):

$$\lambda = p - c \tag{6}$$
$$\dot{\lambda} - r\lambda = c_s q \tag{7}$$
$$\lambda + \mu = 1/d_x \tag{8}$$
$$\dot{\mu} - r\mu = (\lambda + \mu)(-d_f) \tag{9}$$

The first condition states that the shadow price on the resource constraint is the profit on extracting a marginal unit of resource, $p - c$. This restates the first efficiency condition encountered earlier. The second first-order condition reveals the Hotelling rule (which is also known as the **r-percent rule**) when the marginal cost of extraction does not depend on the resource stock s (and thus $c_s = 0$). In that case, the shadow price of the resource constraint must increase at the constant pace determined by the discount rate r (i.e., $\dot{\lambda}/\lambda = r$). Recall that dynamic efficiency requires that the mining operator is indifferent between extracting one unit today or at a future point in time. Differentiating first-order condition (6) with respect to time generates the equation $\dot{\lambda} = \dot{p} - \dot{c}$. Substituting (6), (7) and (3) back into this new equation reveals the time path of the resource price:

$$\frac{\dot{p}}{p} = r - \left[\frac{r + a + b \left[\frac{d}{f} \right] \left[\frac{f-s}{s} \right]}{p/c} \right] \tag{10}$$

The relative change in the profitability of extracting a marginal unit of resource follows an adjusted r-percent rule.

The simple r-percent rule emerges only when the marginal cost of extraction is zero, which is unrealistic. The adjusted r-percent rule in equation (10) states that the rate of price increases of the resource, $\dot{p}/p$, is smaller than the discount rate r, and may even be negative. When the rate of technological progress is fast (a is large), this will tend to reduce $\dot{p}/p$, or even make it negative so that resource prices may fall over time rather. Similar to what described the motion of $\dot{c}/c$, the motion of $\dot{p}/p$ may follow a U-shape over time. At an early stage, resource exploration may be relatively small (d is low), and then $\dot{p}/p$ is positive. When resource exploration is fast (d is high) at later stages, $\dot{p}/p$ may be close to zero or negative. This may be a plateau phase that can persist for considerable time. Eventually, resource scarcity becomes dominant as new discoveries diminish (d approaches zero). Then $\dot{p}/p$ ventures back into positive territory. Taken together, resource prices may experience a U shape with an extended intermediate period when prices are quite stable.

In order to understand the evolution of resource exploration over time, it is again useful to introduce particular functional form for $d(x, s)$. The specification

$$d(x, s) = d_0 \left(\frac{x^g}{s^h} \right) \tag{11}$$

introduces the elasticities $0 < g < 1$ and $h > 0$ for the effect of exploration expenditures on new discoveries and the effect of remaining stock on new discoveries. With $0 < g < 1$, there are diminishing returns to resource exploration. Scarcer resources makes finding new resources harder.

With (11), the third and fourth first-order conditions (8) and (9) of the profit maximum shed light on the evolution of the shadow price of resource exploration. Time-differentiating (8) and making use of the previous equations reveals a time path for resource exploration:

$$\frac{\dot{x}}{x} = \frac{1}{1-g}\left[r - bg\left(\frac{cq}{x}\right)\left(\frac{d}{s}\right)\right] \tag{12}$$

This equation reveals the optimal path of resource exploration. The overall speed of resource exploration depends on the effect of exploration expenditures on new discoveries. Because of diminishing returns to increased exploration expenditures x, the parameter g is between zero and one. Resource exploration is going to be faster in the absence of diminishing returns (when g gets closer to one). Resource exploration is also faster when the discount rate is larger. Thus, "impatient" mining firms (with a high r) engage in more rapid resource exploration than "patient" mining firms (with a low r). The path of resource exploration can follow and inverted-U shape due to the presence of the term with the negative sign after r in equation (12).

The parameters b and g describe the dependence of marginal costs on the remaining resource stock and the elasticity of new discoveries with respect to exploration expenditures. The expression (cq/x) captures the ratio of extraction costs to exploration costs. This expression tends to be large during early stages of the mining operation with little resource scarcity. It will decrease over time as the mining firm will spend more and more on exploration to discover new resources. The ratio of new discoveries to remaining resource stock (d/s) may be relatively stable over time as new discoveries become more expensive and remaining stocks will decline. As a result, the expression in square brackets may be positive at an early stage of the mining operation and may turn negative during late stages of the mining operation: an inverted U-shape.

Lastly, what is the trajectory of resource extraction $q(t)$? Extraction progresses either at a maximal rate or zero. If resource extraction is a competitive business, economic rents will vanish and $q(t)$ will be determined by the path of prices and exploration. When a mining operation has monopolistic power, resource extractions will evolve at a slower pace because the monopolist can extract an economic rent. The monopolist is a resource conservationist!